4.2 换元积分法

4.2.1 第一类换元法（凑微分法）

4.2.2 第二类换元法

基本思路

设 $F'(u)=f(u)$，$u=\varphi(x)$ 可导，则有

$$\mathrm{d}F[\varphi(x)]=f[\varphi(x)]\varphi'(x)\mathrm{d}x$$

$$\therefore \int f[\varphi(x)]\varphi'(x)\mathrm{d}x = F[\varphi(x)]+C$$

$$=F(u)+C\Big|_{u=\varphi(x)}=\int f(u)\,\mathrm{d}u\Big|_{u=\varphi(x)}$$

$$F'(u)=f(u)$$

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x \;\underset{\text{第二类换元法}}{\overset{\text{第一类换元法}}{\rightleftarrows}}\; \int f(u)\mathrm{d}u$$

4.2.1 第一类换元法

定理4. 2. 1. 设 $f(u)$有原函数$F(u)$, $u=\varphi(x)$可导,

则有换元公式

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x = \int f(\varphi(x))\,\mathrm{d}\,\varphi(x)$$

$$= \int f(u)\mathrm{d}u \; = F(u) + C\Big|_{u=\varphi(x)} = F(\varphi(x)) + C$$

(也称配元法, 凑微分法)

例1 求 $\int (ax+b)^m \mathrm{d}x \quad (m \neq -1, a \neq 0).$

解 令 $u = ax+b$,则 $\mathrm{d}u = a\mathrm{d}x$,故

原式 $= \int u^m \frac{1}{a}\mathrm{d}u = \frac{1}{a} \cdot \frac{1}{m+1} u^{m+1} + C$

$$= \frac{1}{a(m+1)}(ax+b)^{m+1} + C$$

注 当 $m=-1$时 $\int \frac{\mathrm{d}x}{ax+b} = \int \frac{1}{u}\frac{1}{a}\mathrm{d}u$

$$= \frac{1}{a}\ln|u| + C = \frac{1}{a}\ln|ax+b| + C$$

例2　求 $\int \frac{dx}{a^2+x^2}\ (a>0)$.

想到公式

$$\int \frac{du}{1+u^2} = \arctan u + C$$

解

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a^2}\int \frac{dx}{1+(\frac{x}{a})^2}$$

令 $u=\frac{x}{a}$, 则 $du=\frac{1}{a}dx$

$$= \frac{1}{a}\int \frac{du}{1+u^2} = \frac{1}{a}\arctan u + C$$

$$= \frac{1}{a}\arctan(\frac{x}{a}) + C \quad = \frac{1}{a}\int \frac{d(\frac{x}{a})}{1+(\frac{x}{a})^2}$$

直接凑?

例3 求 $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}}\ (a>0).$

解 $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \int \frac{\mathrm{d}x}{a\sqrt{1-(\frac{x}{a})^2}} = \int \frac{\mathrm{d}(\frac{x}{a})}{\sqrt{1-(\frac{x}{a})^2}}$

$= \arcsin\frac{x}{a} + C$

想到 $\int \frac{\mathrm{d}u}{\sqrt{1-u^2}} = \arcsin u + C$ 如何计算 $\int f(\varphi(x))\mathrm{d}x$?

$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x = \int f(\varphi(x))\mathrm{d}\varphi(x)$ (直接配元)

例4 求 $\int \frac{\mathrm{d}x}{x^2 - a^2} \quad (a > 0)$

解 $\because \frac{1}{x^2 - a^2} = \frac{1}{2a} \frac{(x+a)-(x-a)}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$

$\therefore$ 原式 $= \frac{1}{2a}\left[\int \frac{\mathrm{d}x}{x-a} - \int \frac{\mathrm{d}x}{x+a}\right]$

$= \frac{1}{2a}\left[\int \frac{\mathrm{d}(x-a)}{x-a} - \int \frac{\mathrm{d}(x+a)}{x+a}\right]$

$= \frac{1}{2a}\left[\ln|x-a| - \ln|x+a|\right] + C = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

例5 求 $\int \tan x \mathrm{d}x$.

解 $\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x} \mathrm{d}x = -\int \frac{\mathrm{d}\cos x}{\cos x}$

$= -\ln|\cos x| + C$

类似地

$\int \cot x \mathrm{d}x = \int \frac{\cos x \mathrm{d}x}{\sin x} = \int \frac{\mathrm{d}\sin x}{\sin x}$

$= \ln|\sin x| + C$

$$\int \frac{f'(x)}{f(x)} \mathrm{d}x = \int \frac{\mathrm{d}(f(x))}{f(x)} = \ln|f(x)| + C$$

常用的几种配元形式:

(1) $\int f(ax+b)\mathrm{d}x = \frac{1}{a}\int f(ax+b)\,\mathrm{d}(ax+b)$

(2) $\int f(x^n)x^{n-1}\,\mathrm{d}x = \frac{1}{n}\int f(x^n)\,\mathrm{d}(x^n)$

(3) $\int f(x^n)\frac{1}{x}\mathrm{d}x = \frac{1}{n}\int f(x^n)\,\frac{1}{x^n}\mathrm{d}(x^n)$

(4) $\int f(x^n)x^{2n-1}\,\mathrm{d}x = \frac{1}{n}\int f(x^n)x^n\,\mathrm{d}x^n$

(5) $\int f(\sin x)\cos x\mathrm{d}x = \int f(\sin x)\mathrm{d}\sin x$

(6) $\int f(\cos x)\sin x\mathrm{d}x = -\int f(\cos x)\,\mathrm{d}\cos x$

(7) $\int f(\tan x)\sec^2 x\mathrm{d}x = \int f(\tan x)\mathrm{d}\tan x$

(8) $\int f(e^x)e^x\mathrm{d}x = \int f(e^x)\,\mathrm{d}e^x$

(9) $\int f(\ln x)\frac{1}{x}\mathrm{d}x = \int f(\ln x)\,\mathrm{d}\ln x$

例6 求 $\int \frac{\mathrm{d}\,x}{x(1+2\ln x)}$.

解 原式 $= \int \frac{\mathrm{d}\ln x}{1+2\ln x} = \frac{1}{2}\int \frac{\mathrm{d}(1+2\ln x)}{1+2\ln x}$

$$= \frac{1}{2}\ln\left|1+2\ln x\right| + C$$

例11 求 $\int \frac{x^3}{(x^2+a^2)^{3/2}}\mathrm{d}x$. $\int f(x^n)x^{2n-1}\mathrm{d}x$

解 原式 $=\frac{1}{2}\int \frac{x^2\,\mathrm{d}(x^2)}{(x^2+a^2)^{3/2}}=\frac{1}{2}\int \frac{(x^2+a^2)-a^2}{(x^2+a^2)^{3/2}}\mathrm{d}(x^2)$

$$=\frac{1}{2}\int (x^2+a^2)^{-1/2}\,\mathrm{d}(x^2)-\frac{a^2}{2}\int (x^2+a^2)^{-3/2}\,\mathrm{d}(x^2)$$

$$=\frac{1}{2}\int (x^2+a^2)^{-1/2}\,\mathrm{d}(x^2+a^2)-\frac{a^2}{2}\int (x^2+a^2)^{-3/2}\,\mathrm{d}(x^2+a^2)$$

$$=\sqrt{x^2+a^2}+\frac{a^2}{\sqrt{x^2+a^2}}+C$$

例10 求 $\int \sec x \mathrm{d}x$.

$$\int \frac{\mathrm{d}x}{x^2-a^2}=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$$

解法1

$$\int \sec x \mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}x = \int \frac{\cos x}{\cos^2 x}\mathrm{d}x = \int \frac{\mathrm{d}\sin x}{\cos^2 x}$$

$$= \int \frac{\mathrm{d}\sin x}{1-\sin^2 x} = \frac{1}{2}\int\left[\frac{1}{1+\sin x}+\frac{1}{1-\sin x}\right]\mathrm{d}\sin x$$

$$= \frac{1}{2}\left[\ln|1+\sin x| - \ln|1-\sin x|\right]+C$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+C$$

解法 2 $(\sec x)' = \sec x \tan x$

$$(\tan x)' = \sec^2 x$$

$$(\sec x + \tan x)' = \sec x(\sec x + \tan x)$$

$$\frac{(\sec x + \tan x)'}{\sec x + \tan x} = \sec x$$ 两边求不定积分？

$$\int \sec x \mathrm{d}x = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \mathrm{d}x$$

$$= \int \frac{\mathrm{d}(\sec x + \tan x)}{\sec x + \tan x} = \ln|\sec x + \tan x| + C$$

思想方法：把分子表示成分母的导数形式

同样可证

$$\int \csc x \mathrm{d}x = \int \frac{\csc x(\csc x - \cot x)}{\csc x - \cot x}\mathrm{d}x$$

$$= \int \frac{\mathrm{d}(\csc x - \cot x)}{\csc x - \cot x}$$

$$= \ln\left|\csc x - \cot x\right| + C$$

$$(\csc x)' = -\csc x \cot x$$

$$(\cot x)' = -\csc^2 x$$

$$(\csc x - \cot x)' = \csc x(\csc x - \cot x)$$

或 $\int \csc x dx = \int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}}$

$$= \int \frac{d\frac{x}{2}}{\tan\frac{x}{2}\cos^2\frac{x}{2}} = \int \frac{d\left(\tan\frac{x}{2}\right)}{\tan\frac{x}{2}} = \ln\left|\tan\frac{x}{2}\right| + C$$

$\because \ln|\csc x - \cot x| + \ln|\csc x + \cot x| = 0$

$$\int \csc x dx = \ln|\csc x - \cot x| + C$$

$$= -\ln|\csc x + \cot x| + C \quad = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C$$

常用基本积分公式(二)

(16) $\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + C$

(17) $\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\frac{x}{a} + C \quad (a>0)$

(18) $\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

(19) $\int \tan x\,\mathrm{d}x = -\ln|\cos x| + C$

(20) $\int \cot x\mathrm{d}x = \ln|\sin x| + C$

(21) $\int \sec x\mathrm{d}x = \ln|\sec x + \tan x| + C$

(22) $\int \csc x\mathrm{d}x = \ln|\csc x - \cot x| + C$

例12 求 $\int \cos^4 x \, dx$.

用倍角公式降幂

解 $\because \quad \cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2$

$$= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$

$$= \frac{1}{4}\left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right)$$

$$= \frac{1}{4}\left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right)$$

$\therefore \quad \int \cos^4 x \, dx = \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$

$$= \frac{1}{4}\left[\frac{3}{2}\int dx + \int \cos 2x \, d(2x) + \frac{1}{8}\int \cos 4x \, d(4x)\right]$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

例8 求$\int \sec^6 x dx$.

解 原式$=\int (\tan^2 x+1)^2 \mathrm{d}\tan x$

$$=\int (\tan^4 x+2\tan^2 x+1)\mathrm{d}\tan x$$

$$=\int (\tan^4 x)\mathrm{d}\tan x+\int 2\tan^2 x\mathrm{d}\tan x+\int \mathrm{d}\tan x$$

$$=\frac{1}{5}\tan^5 x+\frac{2}{3}\tan^3 x+\tan x+C$$

例13 求 $\int \sin^2 x\cos^2 3x\,\mathrm{d}x$.

积化和差

解 $\because$ $\sin^2 x\cos^2 3x = [\frac{1}{2}(\sin 4x - \sin 2x)]^2$

$= \frac{1}{4}\sin^2 4x - \frac{1}{4}\cdot 2\sin 4x\sin 2x + \frac{1}{4}\sin^2 2x$

倍角公式

$= \frac{1}{8}(1-\cos 8x) - \sin^2 2x\cos 2x + \frac{1}{8}(1-\cos 4x)$

$\therefore$ 原式 $= \frac{1}{4}\int \mathrm{d}x - \frac{1}{64}\int \cos 8x\,\mathrm{d}(8x)$

$- \frac{1}{2}\int \sin^2 2x\,\mathrm{d}(\sin 2x) - \frac{1}{32}\int \cos 4x\,\mathrm{d}(4x)$

$= \frac{1}{4}x - \frac{1}{64}\sin 8x - \frac{1}{6}\sin^3 2x - \frac{1}{32}\sin 4x + C$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

例9 求 $\int \frac{\mathrm{d}x}{1+e^x}$.

$$\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln|f(x)| + C$$

解法1 $\int \frac{\mathrm{d}x}{1+e^x} = \int \frac{e^x}{(1+e^x)e^x}\,\mathrm{d}x$

$$= \int \frac{1}{(1+e^x)e^x}\,\mathrm{d}(e^x) = \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right)\mathrm{d}e^x$$

$$= \ln e^x - \ln(1+e^x) + C = x - \ln(1+e^x) + C$$

解法2 $\int \frac{\mathrm{d}x}{1+e^x} = \int \frac{e^{-x}}{1+e^{-x}}\,\mathrm{d}x = -\int \frac{\mathrm{d}(1+e^{-x})}{1+e^{-x}}$

$$= -\ln(1+e^{-x}) + C$$

例9 求 $\int \frac{\mathrm{d}x}{1+e^x}$.

$$\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln|f(x)|+C$$

解法1 $\int \frac{\mathrm{d}x}{1+e^x} = \int \frac{e^x}{(1+e^x)e^x}\mathrm{d}x = x-\ln(1+e^x)+C$

解法2 $\int \frac{\mathrm{d}x}{1+e^x} = \int \frac{e^{-x}}{1+e^{-x}}\mathrm{d}x = -\ln(1+e^{-x})+C$

解法3 $\int \frac{\mathrm{d}x}{1+e^x} = \int \frac{(1+e^x)-e^x}{1+e^x}\mathrm{d}x = \int \mathrm{d}x - \int \frac{\mathrm{d}(1+e^x)}{1+e^x}$

$$= x - \ln(1+e^x)+C$$

$$x-\ln(1+e^x) = \ln e^x - \ln(e^x+1) = \ln(\frac{e^x}{e^x+1}) = \ln(\frac{1}{e^{-x}+1})$$

$$= -\ln(1+e^{-x})$$

三种方法的结果一样

例14 求 $\int \frac{x+1}{x(1+xe^x)}\mathrm{d}x$.

解 原式= $\int \frac{(x+1)\,e^x}{xe^x(1+xe^x)}\mathrm{d}x = \int \frac{1}{xe^x(1+xe^x)}\mathrm{d}(xe^x)$

$$= \int \left(\frac{1}{xe^x} - \frac{1}{1+xe^x}\right)\mathrm{d}(xe^x)$$

$$= \ln\left|xe^x\right| - \ln\left|1+xe^x\right| + C = \ln|x| + x - \ln\left|1+xe^x\right| + C$$

分析: $\mathrm{d}(1+xe^x) = (x+1)e^x\,\mathrm{d}x$

$$\frac{1}{xe^x(1+xe^x)} = \frac{1+xe^x-xe^x}{xe^x(1+xe^x)} = \frac{1}{xe^x} - \frac{1}{1+xe^x}$$

例15 求(1) $\int \frac{\arctan\sqrt{x}}{(1+x)\sqrt{x}}\mathrm{d}x$, (2) $\int \frac{\sin x\cos x}{1+\sin^4 x}\mathrm{d}x$.

解 (1) 原式 $= \int \frac{\arctan\sqrt{x}}{(1+(\sqrt{x})^2)\sqrt{x}}\mathrm{d}x = 2\int \frac{\arctan\sqrt{x}}{(1+(\sqrt{x})^2)}\mathrm{d}\sqrt{x}$

$$= 2\int \arctan\sqrt{x}\ \mathrm{d}(\arctan\sqrt{x}) = (\arctan\sqrt{x})^2 + C$$

(2) 原式 $= \int \frac{\sin x}{1+(\sin^2 x)^2}\mathrm{d}(\sin x)$

$$= \frac{1}{2}\int \frac{1}{1+(\sin^2 x)^2}\mathrm{d}(\sin^2 x) = \frac{1}{2}\arctan(\sin^2 x) + C$$

$\frac{1}{2}\cdot\frac{1}{\sqrt{x}}dx = d\sqrt{x}$ $\quad 2\sin x\cos x dx = d\sin^2 x$

小结　常用简化技巧:

(1) 分项积分：利用积化和差;分式分项;

$$1=\sin^2 x+\cos^2 x \text{等}$$

(2) 降低幂次：利用倍角公式 ，如

$$\cos^2 x=\frac{1}{2}(1+\cos 2x);\qquad \sin^2 x=\frac{1}{2}(1-\cos 2x);$$

凑幂法

$$\begin{cases}\int f(x^n)x^{n-1}\,\mathrm{d}x=\frac{1}{n}\int f(x^n)\,\mathrm{d}\,x^n\\ \int f(x^n)\frac{1}{x}\,\mathrm{d}x=\frac{1}{n}\int f(x^n)\frac{1}{x^n}\,\mathrm{d}\,x^n\end{cases}$$

(3) 统一函数:利用三角公式;配元方法

(4) 巧妙换元或配元

思考与练习

1 下列各题求积分的方法有何不同?

(1) $\int \frac{dx}{4+x} = \int \frac{d(4+x)}{4+x}$

(2) $\int \frac{dx}{4+x^2} = \frac{1}{2}\int \frac{d(\frac{x}{2})}{1+(\frac{x}{2})^2}$

(3) $\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{d(\frac{x}{2})}{\sqrt{1-(\frac{x}{2})^2}}$

(4) $\int \frac{x}{4+x^2}dx = \frac{1}{2}\int \frac{d(4+x^2)}{4+x^2}$

(5) $\int \frac{x^2}{4+x^2}dx = \int [1-\frac{4}{4+x^2}]d\,x$

(6) $\int \frac{dx}{4-x^2} = \frac{1}{4}\int [\frac{1}{2-x}+\frac{1}{2+x}]dx$

(7) $\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{d(x-2)}{\sqrt{4-(x-2)^2}}$

(8) $\int \sqrt{4-x^2}\,dx$

(9,10) $\int \frac{1}{\sqrt{x^2 \pm 4}}dx$

(11) $\int \sqrt{4+x^2}\,dx$

(12) $\int \sqrt{x^2-4}\,dx$

4.2.2 第二类换元法

第一类换元法解决的问题

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x = \int f(u)\mathrm{d}u \Big|_{u=\varphi(x)}$$

难求　　　　　易求

若所求积分 $\int f(u)\mathrm{d}u$ 难求，

$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x$ 易求，

则得第二类换元积分法.

常用的第二类换元法有

三角代换，倒代换，根式代换

定理4.2.2 设 $x=\psi(t)$ 是单调可导函数，且 $\psi'(t)\neq 0$,

$f[\psi(t)]\psi'(t)$ 具有原函数，则有换元公式

$$\int f(x)\mathrm{d}x=\int f[\psi(t)]\psi'(t)\mathrm{d}t\Big|_{t=\psi^{-1}(x)}$$

其中 $t=\psi^{-1}(x)$ 是 $x=\psi(t)$ 的反函数.

$$\int f(x)\mathrm{d}x$$

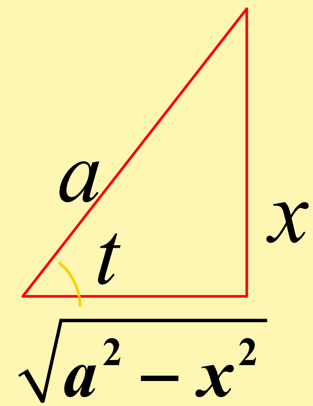
$$\Big\| \quad \text{令 } x=\psi(t)$$

$$\int f[\psi(t)]\psi'(t)\mathrm{d}t\Big|_{t=\psi^{-1}(x)}$$

例16 求 $\int\sqrt{a^2-x^2}\,\mathrm{d}x\ \ (a>0)$.

解 令 $x=a\sin t,\ t\in(-\frac{\pi}{2},\frac{\pi}{2})$, 则 $\mathrm{d}x=a\cos t\,\mathrm{d}t$

$$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2 t}=a\cos t$$



$$\therefore \text{原式}=\int a\cos t\cdot a\cos t\,\mathrm{d}t=a^2\int\cos^2 t\,\mathrm{d}t$$

$$=a^2\int\frac{1+\cos 2t}{2}\mathrm{d}t$$

$$=a^2\left(\frac{t}{2}+\frac{\sin 2t}{4}\right)+C$$

$t\in(0,\frac{\pi}{2})$时

$$\sin 2t=2\sin t\cos t=2\cdot\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}$$

$$=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C$$

例17 求 $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} \quad (a>0).$

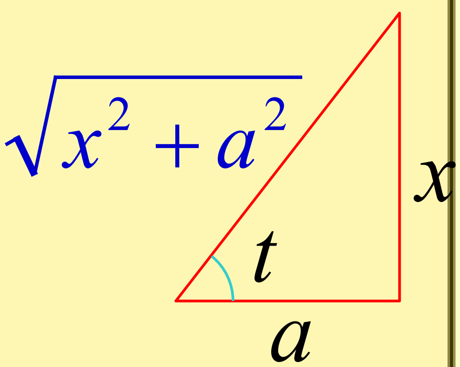
解 令 $x = a\tan t,\ t\in(-\frac{\pi}{2},\frac{\pi}{2})$, 则

$$\sqrt{x^2+a^2}=\sqrt{a^2\tan^2 t+a^2}=a\sec t \qquad \mathrm{d}x=a\sec^2 t\,\mathrm{d}t$$

$$\therefore \text{原式}=\int\frac{a\sec^2 t}{a\sec t}\mathrm{d}t=\int\sec t\,\mathrm{d}t$$

$$=\ln\left|\sec t+\tan t\right|+C_1$$

$$=\ln\left[\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right]+C_1$$



$$=\ln\left[x+\sqrt{x^2+a^2}\right]-\ln a+C_1$$

$t\in(0,\frac{\pi}{2})$时

$$=\ln\left[x+\sqrt{x^2+a^2}\right]+C \qquad (C=C_1-\ln a)$$

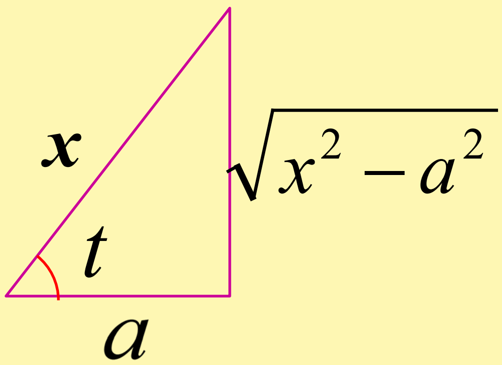
例18 求 $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}\ (a>0).$

解 当$x>a$时，令 $x=a\sec t,\ t\in(0,\frac{\pi}{2})$，则

$$\sqrt{x^2-a^2}=\sqrt{a^2\sec^2 t-a^2}=a\tan t$$

$$\mathrm{d}x=a\sec t\tan t\,\mathrm{d}t$$

$$\therefore \text{原式}=\int\frac{a\sec t\tan t}{a\tan t}\mathrm{d}t=\int\sec t\,\mathrm{d}t$$



$$=\ln\left|\sec t+\tan t\right|+C_1$$

$$=\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C_1$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|+C\qquad(C=C_1-\ln a)$$

当 $x<-a$ 时, 令 $x=-u$, 则 $u>a$,于是

$$\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=-\int\frac{\mathrm{d}u}{\sqrt{u^2-a^2}}=-\ln\left|u+\sqrt{u^2-a^2}\right|+C_1$$

$$=-\ln\left|-x+\sqrt{x^2-a^2}\right|+C_1$$

$$=\ln\left|\left(-x+\sqrt{x^2-a^2}\right)^{-1}\right|+C_1$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|+C\quad(C=C_1-2\ln a)$$

$x>a$ 时, $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=\ln\left|x+\sqrt{x^2-a^2}\right|+C$

说明(1) 以上几例所使用的均为三角代换.

三角代换的目的是化掉根式.

一般规律如下：当被积函数中含有

(1) $\sqrt{a^2-x^2}$ 可令 $x=a\sin t;\quad t\in(-\frac{\pi}{2},\frac{\pi}{2})$

(2) $\sqrt{a^2+x^2}$ 可令 $x=a\tan t;\quad t\in(-\frac{\pi}{2},\frac{\pi}{2})$

(3) $\sqrt{x^2-a^2}$ 可令 $x=a\sec t.$

$x>a$时, $t\in(0,\frac{\pi}{2})$ $x<-a$时，$t\in(\frac{\pi}{2},\pi)$

或当$x<-a$时, 令$x=-u,\cdots$

含$\sqrt{ax^2+bx+c}$的积分

通过配方，可化为上面三种情况之一。

例19.求$\int \frac{\mathrm{d}x}{\sqrt{1+x-x^2}}$.

解: 原式 $= \int \frac{\mathrm{d}(x-\frac{1}{2})}{\sqrt{(\frac{\sqrt{5}}{2})^2-(x-\frac{1}{2})^2}} = \arcsin\frac{2x-1}{\sqrt{5}}+C$

例 $\int \frac{dx}{\sqrt{1+x+x^2}}$

$$= \int \frac{dx}{\sqrt{(\frac{\sqrt{3}}{2})^2+(x+\frac{1}{2})^2}} = \int \frac{d(x+\frac{1}{2})}{\sqrt{(\frac{\sqrt{3}}{2})^2+(x+\frac{1}{2})^2}}$$

例19 求 $\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x\ (a>0)$ 令 $x=a\sin t, t\in(-\frac{\pi}{2},\frac{\pi}{2}), t\neq 0$

解 令 $x=\frac{1}{t}$，则 $\mathrm{d}x=\frac{-1}{t^2}\mathrm{d}t$

当 $x>0$ 时，

$$原式 =\int \frac{\sqrt{a^2-\frac{1}{t^2}}}{\frac{1}{t^4}}\cdot\frac{-1}{t^2}\mathrm{d}t=-\int (a^2t^2-1)^{\frac{1}{2}}|t|\mathrm{d}t$$

$$原式 =-\frac{1}{2a^2}\int (a^2t^2-1)^{\frac{1}{2}}\mathrm{d}(a^2t^2-1)$$

$$=-\frac{(a^2t^2-1)^{\frac{3}{2}}}{3a^2}+C=-\frac{(a^2-x^2)^{\frac{3}{2}}}{3a^2x^3}+C$$

当 $x<0$ 时，令 $u=-x,\cdots\cdots$

例21 $\int \frac{1}{\sqrt{1+e^x}}dx$

解 令$\sqrt{1+e^x}=t>1$,则$1+e^x=t^2$,$e^x dx=2tdt$,

从而$dx=\frac{2t}{t^2-1}dt$,

原式$=\int \frac{2}{t^2-1}dt=\ln\left|\frac{t-1}{t+1}\right|+C=\ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right|+C$

解: $\int \frac{1}{\sqrt{1+e^x}}dx=\int \frac{e^{-x}}{\sqrt{e^{-x}+e^{-2x}}}dx$

$=-\int \frac{d(e^{-x})}{\sqrt{e^{-x}}\cdot\sqrt{1+e^{-x}}}=-\int \frac{d(\sqrt{e^{-x}})}{2\sqrt{1+e^{-x}}}=\cdots$

小结:

1. 第二类换元法常见类型:

(1) $\int f(x, \sqrt[n]{ax+b})\,\mathrm{d}x$，令 $t = \sqrt[n]{ax+b}$

(2) $\int f(x, \sqrt[n]{\frac{ax+b}{cx+d}})\,\mathrm{d}x$，令 $t = \sqrt[n]{\frac{ax+b}{cx+d}}$

} 第四节讲

如: $\int \frac{x^5}{\sqrt{1+x^2}}\,\mathrm{d}x$

(3) $\int f(x, \sqrt{a^2 - x^2})\,\mathrm{d}x$，令 $x = a\sin t$ 或 $x = a\cos t$

(4) $\int f(x, \sqrt{a^2 + x^2})\,\mathrm{d}x$，令 $x = a\tan t$

(5) $\int f(x, \sqrt{x^2 - a^2})\,\mathrm{d}x$，令 $x = a\sec t$

(6) $\int f(a^x)\mathrm{d}x$，令 $t=a^x$

(7) 分母中因子次数较高时，可试用**倒代换**

如：$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+a^2}}$，　　$\int \frac{\mathrm{d}x}{x(x^7+2)}$

2. 常用基本积分公式(三)

(23, 24) $$\int \frac{1}{\sqrt{x^2 \pm a^2}}\mathrm{d}x = \ln\left|x+\sqrt{x^2 \pm a^2}\right| + C$$

(25) $$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$$